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A study of nonlinear wave resistance using integral equations in Fourier space

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An attempt is made in this paper to tackle the problem of nonlinear wave resistance by formulating it in Fourier space and by deriving a nonlinear integral equation for the wave amplitude by an approach similar to the one leading to the Zakharov equation.

The procedure is illustrated for two simple examples of two- and three-dimensional travelling pressure distributions. A regular perturbation solution up to third-order terms in the slenderness parameter shows that the expansion is not uniform for small Froude numbers. A uniform, generalized, expansion is then constructed, with its leading term satisfying a new nonlinear integral equation. This rather simple integral equation, of a Volterra type, is solved numerically. The generalized wave drag is shown to be significantly larger than the one predicted by the regular perturbation expansion at small Froude number. The method adopted here has the advantage of singling out in a systematic manner the terms of the free-surface conditions which cause the small-Froude-number non-uniformity, and it is applicable to both two- and three-dimensional flows. The results are compared with existing approximate methods of computing wave drag at low Froude numbers. It is found that quasilinearized approximations may be quite accurate for the examples considered here.

1. Introduction

For many decades the study of the wave resistance of steadily moving bodies or pressure distributions has been dominated by linear theory. The starting point of this period is considered to be marked by the pioneering works of Kelvin (1887) and Michell (1898) (a review of the various stages of development of the theory of wave resistance can be found in Tulin 1978). It is only in the last two decades that nonlinear effects have been studied with increasing intensity by naval hydrodynamicists. These studies have been motivated by the lack of agreement between the linearized theory and measurements on one hand, and have been made possible by the advent of fast electronic computers on the other.

The linearized approximate solution can be obtained in a rational manner by regarding it as a first-order term in an asymptotic expansion in a slenderness or deep-submergence small parameter (e.g. Wehausen & Laitone 1960). The natural extension into the nonlinear domain is achieved by calculating the second-order approximation, which is quadratic in the small parameter. This avenue has not been found so far to be promising for two reasons: first, the computations become very tedious, even by numerical means, and, secondly, and more importantly, the expansion becomes non-uniform for small Froude numbers. The presence of a third lengthscale besides those characterizing the body, namely the wavelength of the far free waves, is an omen for non-uniformity in many problems of fluid mechanics (Van Dyke 1964).

A few approximate approaches have been developed in the past in order to circumvent these difficulties, and a lucid and comprehensive discussion, which is beyond the scope of the present study, may be found in Tulin (1978). In essence, significant progress has been achieved in the understanding of small-Froude-number nonlinear effects in two-dimensional flows. This progress was made possible by the relative simplicity of two-dimensional flows, which permits one to use the powerful tool of the theory of analytic functions of complex variables. On this basis, a few studies (e.g. Ogilvie 1968; Dagan 1975; Doctors & Dagan 1980) have arrived at the view that the small-Froude-number nonlinear effect stems from the interaction between the free waves generated by the body or the pressure on one hand and the slowly varying non-uniform flow prevailing near the disturbances on the other. A few quasilinearization techniques, discussed by Doctors & Dagan (1980) for a particular example and by Tulin (1978) in a general manner, have been suggested in order to account for this effect.

Less progress has been achieved in the case of three-dimensional flows, which are the ones of paramount interest in applications. Some quasilinearization techniques have been suggested as well, and two of them, those of Inui & Kajitani (1977) and Dawson (1977), will be discussed in §6. Their validation has been confined, however, to using them in order to solve by numerical methods a few examples of flow past ship forms and comparing the computed results for the wave drag with measurements. Although the ultimate test for any theory is its degree of agreement with experimental results, we believe that a better understanding of the mechanism of three-dimensional wave generation on theoretical grounds is an important step still to be accomplished. The present study has been motivated by this need, and its primary aim is to derive uniform, nonlinear solutions of the wave-generation problem by theoretical means and to compare them with existing, approximate techniques. To achieve this goal we have investigated the solution of the free-surface flow problem in Fourier space. This line of attack has been motivated by the considerable progress attained in the understanding of nonlinear interaction of free-surface waves by the study of the Zakharov (1968) equation, which is formulated in Fourier space. Furthermore, unlike complex variables, the Fourier-transform methodology applies equally to two- and three-dimensional flows. The present study presents only a first step in this direction, since it applies the approach to the wave resistance of a particular case of a travelling pressure distribution, while a similar study for a submerged cylinder or sphere has been published elsewhere (Dagan & Miloh 1985).

The plan of the paper is as follows. In §2 we cast the equations of free-surface flow as an integral equation in Fourier space along the lines of Zakharov's (1968) approach. This equation is first solved by the classical small-perturbation expansion approximation, and in §§3 and 4 we derive in a closed, analytical, form the first- and second-order solutions for a particular travelling pressure distribution. These solutions reveal in a clear fashion the presence of the small-Froude-number non-uniformity and its origin. The main contribution of the present study is made in §5, where we present an integral equation which renders a nonlinear uniformly valid solution in a domain of small Froude numbers, for which the perturbation expansion fails. This integral equation, of a Volterra type, is much simpler to handle than the original one and can be easily solved numerically. Finally, in §6 we compare the new solution with the ones based on various quasilinearized approximations, including those already mentioned. In Appendix B we show that for the range of values considered in the selected examples of pressure distributions the nonlinear third-order effects are relatively small.

2. Mathematical statement of the problem

We consider the irrotational flow of a heavy fluid caused by a pressure distribution acting on the free surface and which moves steadily in the x-direction. Variables are made dimensionless with respect to the velocity U', the lengthscale U'^2/g and the fluid specific mass ρ'. The flow is referred to a Cartesian system attached to the moving pressure, with the x- and z-coordinates in the horizontal plane of the unperturbed free surface and y a vertical coordinate pointing upwards. The unperturbed velocity vector therefore has components $(-1, 0, 0)$ in this system. With Φ the disturbance velocity potential and $y = \eta(x, z)$ the equation of the free surface, the exact equations governing the flow are

$$\nabla^2\Phi + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad (y \leqslant \eta(x, z)), \tag{2.1}$$

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial x} - \eta - \frac{1}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2 - \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi &= p(x, z), \\ \frac{\partial\Phi}{\partial y} + \frac{\partial\eta}{\partial x} - \nabla\eta\cdot\nabla\Phi &= 0 \end{aligned}\right\} \quad (y = \eta(x, z)). \tag{2.2}$$

where $p(x, z)$ is a prescribed pressure acting on the free surface and $\nabla \equiv (\partial/\partial x, \partial/\partial z)$ denotes the horizontal gradient operator. These equations have to be supplemented by the radiation condition, which requires that waves propagate downstream.

Following Zakharov (1968), we define

$$\Psi(x, z) = \Phi(x, \eta(x, z), z) \quad \text{and} \quad \chi(x, z) = \frac{\partial\Phi}{\partial y}(x, \eta(x, z), z)$$

and rewrite the free-surface boundary conditions (2.2) by chain differentiation as

$$\left.\begin{aligned} &\frac{\partial\Psi}{\partial x} - \eta + \tfrac{1}{2}\chi^2 - \tfrac{1}{2}\nabla\Psi\cdot\nabla\Psi + \tfrac{1}{2}\chi^2\,\nabla\eta\cdot\nabla\eta = p(x, z), \\ &\chi + \frac{\partial\eta}{\partial x} - \nabla\eta\cdot\nabla\Psi + \chi\,\nabla\eta\cdot\nabla\Psi = 0. \end{aligned}\right\} \tag{2.3}$$

The next step is to take the Fourier transform (FT) of (2.3), and for this purpose let us first define the FT of $\Psi(x, z)$ and $\eta(x, z)$ by

$$\left.\begin{aligned} \psi(\boldsymbol{k}) &= (2\pi)^{-\frac{1}{2}m}\int_{-\infty}^{\infty}\Psi(x, z)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}, \\ \zeta(\boldsymbol{k}) &= (2\pi)^{-\frac{1}{2}m}\int_{-\infty}^{\infty}\eta(x, z)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}, \end{aligned}\right\} \tag{2.4}$$

where $m = 1$ or 2 for two- and three-dimensional flows respectively. Here $\boldsymbol{x}$ denotes a coordinate vector with components (x, z), $\boldsymbol{k}$ is the wavenumber vector with components (k_x, k_z) and integration is carried out in the (x, z)- and (k_x, k_z)-planes respectively. For three-dimensional flows we operate with the polar coordinates defined by $k_x = \rho\cos\theta$ and $k_z = \rho\sin\theta$. In the case of two-dimensional flow the coordinate z has to be suppressed in all equations, and $\boldsymbol{x}$ and $\boldsymbol{k}$ should be replaced by k and x respectively. It should be emphasized that $\psi(k)$ (2.4) differs from $\phi(\boldsymbol{k})$, the conventional definition of the FT of $\Phi(x, 0, z)$, the velocity potential on a horizontal plane. In fact the two are related by

$$\left.\begin{aligned} \Psi(x,z) &= (2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} \phi(\boldsymbol{k})\, \mathrm{e}^{|\boldsymbol{k}|\eta(x,z)}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{k}, \\ \chi(x,z) &= (2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} |\boldsymbol{k}|\, \phi(\boldsymbol{k})\, \mathrm{e}^{|\boldsymbol{k}|\eta(x,z)}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{k}, \end{aligned}\right\} \tag{2.5}$$

which results from taking the FT of the Laplace equation (2.1).

Applying the FT to (2.3), with convolution integrals resulting from FT of products, yields a set of two integral equations for the FT of Ψ, η and χ. In Zakharov's procedure the latter is expressed in terms of the first by invoking weak nonlinearity and expanding $\exp(|\boldsymbol{k}|\eta)$ in (2.5) in a Taylor series to third order in $|\boldsymbol{k}|\eta$. Eventually, by using an iterative procedure, the FT of $\chi(\boldsymbol{x})$ is expressed with the aid of $\psi(\boldsymbol{k})$ and $\zeta(\boldsymbol{k})$. The details of this derivation are not presented here, and the interested reader is referred to the original work of Zakharov (1968) or to the comprehensive review of Yuen & Lake (1982). The final results for the FT of (2.3) are the following Zakharov-type coupled integral equations, correct to third order in the amplitude ζ:

$$\begin{aligned} -\mathrm{i}k_x\, \zeta(\boldsymbol{k}) + |\boldsymbol{k}|\, \psi(\boldsymbol{k}) &= (2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} K_1(\boldsymbol{k},\boldsymbol{k}_1)\, \psi(\boldsymbol{k}_1)\, \zeta(\boldsymbol{k}-\boldsymbol{k}_1)\, \mathrm{d}\boldsymbol{k}_1 \\ &\quad + (2\pi)^{-\frac{1}{2}m} \iint_{-\infty}^{\infty} K_2(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\, \psi(\boldsymbol{k}_1)\, \zeta(\boldsymbol{k}_2)\, \zeta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathrm{i}k_x\, \psi(\boldsymbol{k}) + \zeta(\boldsymbol{k}) &= -\Pi(\boldsymbol{k}) + (2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} K_3(\boldsymbol{k},\boldsymbol{k}_1)\, \psi(\boldsymbol{k}_1)\, \psi(\boldsymbol{k}-\boldsymbol{k}_1)\, \mathrm{d}\boldsymbol{k}_1 \\ &\quad + (2\pi)^{-\frac{1}{2}m} \iint_{-\infty}^{\infty} K_4(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\, \psi(\boldsymbol{k}_1)\, \psi(\boldsymbol{k}_2)\, \zeta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2. \end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7) may be also reproduced from Yuen & Lake (1982, equations (103)–(104)), where one can find also a more detailed derivation. The time derivatives in the original Zakharov equations there have been replaced here by $\mathrm{i}k_x$ and a typographical error in the last term of (2.7) has been also corrected. The major, and essential, difference, however, between the above formulation and that leading to the so-called Zakharov integral equation is in the appearance of Π, the FT of the pressure p, as a forcing term in the right-hand side of (2.7).

The kernels K_1, K_2, K_3 and K_4 have the following expressions (Yuen & Lake 1982)

$$\left.\begin{aligned} K_1(\boldsymbol{k},\boldsymbol{k}_1) &= |\boldsymbol{k}|\,|\boldsymbol{k}_1| - \boldsymbol{k}\cdot\boldsymbol{k}_1, \\ K_2(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) &= \tfrac{1}{2}|\boldsymbol{k}|\,|\boldsymbol{k}_1|\,[|\boldsymbol{k}|+|\boldsymbol{k}_1|-2|\boldsymbol{k}-\boldsymbol{k}_1|], \\ K_3(\boldsymbol{k},\boldsymbol{k}_1) &= \tfrac{1}{2}[|\boldsymbol{k}_1|\,|\boldsymbol{k}-\boldsymbol{k}_1|+\boldsymbol{k}_1\cdot(\boldsymbol{k}-\boldsymbol{k}_1)], \\ K_4(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) &= |\boldsymbol{k}_1|\,|\boldsymbol{k}_2|\,[|\boldsymbol{k}_1|-|\boldsymbol{k}-\boldsymbol{k}_1|]. \end{aligned}\right\} \tag{2.8}$$

At this point the present analysis deviates from the common procedure leading to the Zakharov equation, which considers the nonlinear third-order interaction between a rapidly and a slowly varying wave.

(*a*) (*b*)

FIGURE 1. The integration path for the inversion of the Fourier transforms: (*a*) two-dimensional flow; (*b*) three-dimensional flow.

Elimination of ψ from (2.6) and (2.7) and retaining terms up to third order yields a single integral equation for the FT of the free surface profile

$$A(\boldsymbol{k})\,\zeta(\boldsymbol{k}) = -\Pi(\boldsymbol{k}) + (2\pi)^{-\frac{1}{2}m}\int_{-\infty}^{\infty} P_2(\boldsymbol{k},\boldsymbol{k}_1)\,\zeta(\boldsymbol{k}_1)\,\zeta(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1$$
$$+ (2\pi)^{-\frac{1}{2}m}\iint_{-\infty}^{\infty} P_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,\zeta(\boldsymbol{k}_1)\,\zeta(\boldsymbol{k}_2)\,\zeta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2, \quad (2.9)$$

where the function $A(\boldsymbol{k})$ is given by

$$A(\boldsymbol{k}) = 1 - \frac{k_x^2}{|\boldsymbol{k}|} \quad (2.10)$$

and $A(\boldsymbol{k}) = 0$ has zeros at $k_I = 1$ and $k_{II} = -1$ in two dimensions and at $\rho = \sec^2\theta$ in three dimensions. The radiation condition is automatically satisfied if the integration path in the inversion of ζ in the complex plane in which k or ρ serve as real axes, respectively, circumvents the zeros of A from below (figure 1). The kernels P_2 and P_3 in (2.9) are given by

$$P_2(\boldsymbol{k},\boldsymbol{k}_1) = \frac{k_{1x}}{|\boldsymbol{k}|\,|\boldsymbol{k}_1|}\left[k_x K_1(\boldsymbol{k},\boldsymbol{k}_1) - \frac{|\boldsymbol{k}|\,(k_x-k_{1x})}{|\boldsymbol{k}-\boldsymbol{k}_1|}K_3(\boldsymbol{k},\boldsymbol{k}_1)\right], \quad (2.11)$$

$$P_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{k_{1x}}{|\boldsymbol{k}|\,|\boldsymbol{k}_1|}\left[k_x K_2(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) - \frac{|\boldsymbol{k}|\,k_{2x}}{|\boldsymbol{k}_2|}K_4(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\right.$$
$$+\frac{k_x}{|\boldsymbol{k}_1+\boldsymbol{k}_2|}K_1(\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2)\,K_1(\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_1)$$
$$\left.-\frac{2(k_x-k_{1x}-k_{2x})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|\,|\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2|}K_3(\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2)\,K_1(\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_1)\right]. \quad (2.12)$$

Finally, it was found convenient to define the complex amplitude by

$$a(\boldsymbol{k}) = -A(\boldsymbol{k})\,\zeta(\boldsymbol{k})/\Pi(\boldsymbol{k}), \quad (2.13)$$

which after substitution in (2.9) leads to the following integral equation:

$$a(\boldsymbol{k}) = 1 + (2\pi)^{-\frac{1}{2}m}\int_{-\infty}^{\infty} Q_2(\boldsymbol{k},\boldsymbol{k}_1)\,a(\boldsymbol{k}_1)\,a(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1$$
$$+ (2\pi)^{-m}\iint_{-\infty}^{\infty} Q_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,a(\boldsymbol{k}_1)\,a(\boldsymbol{k}_2)\,a(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2, \quad (2.14)$$

with the associated kernels given by

$$\left.\begin{aligned} Q_2(\boldsymbol{k},\boldsymbol{k}_1) &= -\frac{P_2(\boldsymbol{k},\boldsymbol{k}_1)\,\Pi(\boldsymbol{k}_1)\,\Pi(\boldsymbol{k}-\boldsymbol{k}_1)}{A(\boldsymbol{k}_1)A(\boldsymbol{k}-\boldsymbol{k}_1)\,\Pi(\boldsymbol{k})},\\ Q_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) &= \frac{P_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,\Pi(\boldsymbol{k}_1)\,\Pi(\boldsymbol{k}_2)\,\Pi(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)}{A(\boldsymbol{k}_1)A(\boldsymbol{k}_2)A(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\Pi(\boldsymbol{k})}.\end{aligned}\right\} \quad (2.15)$$

The nonlinear integral equation (2.14) is the starting point of the present study, and its solution is our main aim. Once (2.14) is solved, the free-surface profile is given by

$$\eta(x,z) = -(2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} \frac{a(\boldsymbol{k})\,\Pi(\boldsymbol{k})}{A(\boldsymbol{k})}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k}, \tag{2.16}$$

where the integration path is indented as indicated before (figure 1). The far free waves are obtained from (2.16) by letting $x \to -\infty$. In the case of three-dimensional flows they result from the pole $\rho = \sec^2\theta$ and are given in terms of the complex amplitude $a(\rho,\theta)$ by

$$\eta(x,z) = \tfrac{1}{2}i \int_{-\pi}^{\pi} a(\sec^2\theta,\theta)\,\Pi(\sec^2\theta,\theta)\, e^{-i\sec^2\theta(x\cos\theta + z\sin\theta)}\sec^4\theta\, d\theta. \tag{2.17}$$

Following Havelock (1934), the wave resistance (made dimensionless with respect to $\rho' U'^6/g^2$) is found by evaluating the far free-wave momentum flux, resulting in

$$D = \tfrac{1}{2}\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} a(\sec^2\theta,\theta)\,a^*(\sec^2\theta,\theta)\,\Pi(\sec^2\theta,\theta)\,\Pi^*(\sec^2\theta,\theta)\sec^5\theta\, d\theta, \tag{2.18}$$

with the asterisk indicating complex conjugate.

In the two-dimensional case the far free waves are Stokes waves, which have the general expression

$$\eta(x) = \eta^{(1)}\, e^{ix} + \eta^{(1)*}\, e^{-ix} + \eta^{(2)}\, e^{2ix} + \eta^{(2)*}\, e^{-2ix} + \dots, \tag{2.19}$$

where the coefficients $\eta^{(1)}, \eta^{(2)}, \dots$ are obtained from the contributions of the poles related to A and a in (2.16). The wave resistance (see Appendix A) is given to third-order terms in the amplitude by

$$D = \eta^{(1)}\eta^{(1)*}. \tag{2.20}$$

If we assume that the pressure distribution is proportional to a small slenderness parameter $\epsilon = o(1)$ we may seek an asymptotic solution of (2.14) for $\Pi(\boldsymbol{k}) = o(1)$. The classical approach is to expand $a(\boldsymbol{k})$ in a regular perturbation-expansion power series in ϵ. Thus, with

$$a(\boldsymbol{k}) = a_1(\boldsymbol{k}) + a_2(\boldsymbol{k}) + a_3(\boldsymbol{k}) + \dots \tag{2.21}$$

substituted in (2.14), separating terms of the same order yields, after a few manipulations,

$$a_1(\boldsymbol{k}) = 1, \tag{2.22}$$

$$a_2(\boldsymbol{k}) = (2\pi)^{-\frac{1}{2}m} \int_{-\infty}^{\infty} Q_2(\boldsymbol{k},\boldsymbol{k}_1)\, d\boldsymbol{k}_1, \tag{2.23}$$

$$a_3(\boldsymbol{k}) = (2\pi)^{-m} \iint_{-\infty}^{\infty} Q_3'(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2, \tag{2.24}$$

where

$$Q_3'(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = Q_3(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) + Q_2(\boldsymbol{k},\boldsymbol{k}_1)\,Q_2(\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_2) + Q_2(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1)\,Q_2(\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_2). \tag{2.25}$$

In the regular perturbation solution each term is expressed explicitly as a function of terms of lower order. Once a_1, a_2, ..., are determined from (2.22), (2.23), ..., the far free waves and the wave drag may be evaluated with the aid of (2.16)–(2.20). This will be demonstrated in §§3 and 4 for two simple examples of pressure distributions.

The first approximation in (2.13), for $a_1 = 1$ (2.22), i.e.

$$\zeta_1(\boldsymbol{k}) = -\Pi(\boldsymbol{k})/A(\boldsymbol{k}), \tag{2.26}$$

is the FT of the classical linearized solution of the free-surface problem. Similarly, ζ_2 and ζ_3 can be obtained by first expanding the equations of flow and subsequently taking the FT of the equations of various orders.

3. Illustration of second-order regular perturbation solution (two-dimensional flow)

As a first step toward the investigation of the solution of the integral equation (2.14), we have considered the following example of a two-dimensional pressure distribution

$$p(x) = \frac{\epsilon l}{1+(x/l)^2}, \quad \epsilon = \frac{p'_{\max}}{\rho' g l'}, \quad W = \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = \pi\epsilon l^2. \tag{3.1}$$

where $l = Fr^{-2}$ is the dimensionless lengthscale of the pressure patch (Fr is the Froude number), ϵ is the small parameter reflecting its slenderness and W is the dimensionless weight supported by the distribution.

The choice of this particular pressure distribution has originated from the study of the nonlinear wave resistance of a submerged cylinder, which is the object of a separate article (Dagan & Miloh 1985). Its relative simplicity allows us to derive closed-form solutions and to grasp the main effect of nonlinearity at the expense of simple computations. At any rate, our main interest resides in three-dimensional flows, and it is worthwhile to consider more complex distributions only for such flows. The study of the planar flow is still of theoretical interest for the purpose of comparing it with existing solutions obtained by other methods. Furthermore, as we shall show in §4, it facilitates the solution of the three-dimensional problem.

The FT of the pressure p (3.1) is given by

$$\Pi(k) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\epsilon l^2\,\mathrm{e}^{-|k|\,l}. \tag{3.2}$$

Our interest resides in the wave resistance generated by the moving pressure p. The first-order far free waves are readily found from the solution $\zeta_1(\boldsymbol{k})$ (2.26), by its substitution in the two-dimensional version of (2.16) and extraction of the contributions from the poles k_{I} and k_{II}, the only surviving terms for $x \to -\infty$. The general result is

$$\eta_1 = \mathrm{i}(2\pi)^{\frac{1}{2}}[\Pi(1)\,\mathrm{e}^{\mathrm{i}x} - \Pi(-1)\,\mathrm{e}^{-\mathrm{i}x}]. \tag{3.3}$$

The first-order term of the dimensionless wave resistance is found immediately from (2.20), (2.22) and (3.3) as follows

$$D_1 = 2\pi a_1(1)\,a_1^*(-1)\,\Pi(1)\,\Pi(-1) = 2\pi\,\Pi^2(1), \tag{3.4}$$

which yields for the particular case (3.2)

$$D_1 = (\pi\epsilon l^2)^2\,\mathrm{e}^{-2l}, \quad \mathscr{D}_1 = \frac{D_1}{W} = \pi\epsilon l^2\,\mathrm{e}^{-2l}, \tag{3.5}$$

where $\mathscr{D}_1$ is the drag-over-weight ratio, a meaningful physical quantity for pressures that support a vehicle. The dependence of $\mathscr{D}_1/\epsilon$ upon the Froude number $Fr = l^{-\frac{1}{2}}$ is depicted in figure 2.

The second-order nonlinear term is given explicitly by (2.23) for $m = 1$. Substitution

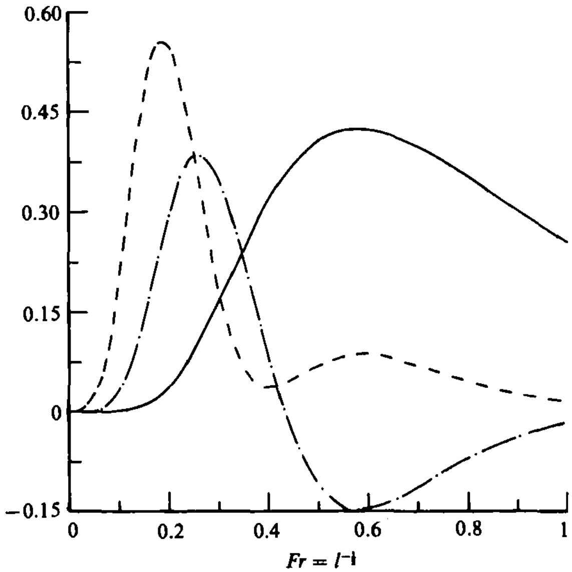

FIGURE 2. The wave-drag components based on the second-order regular perturbation expansion of the wave amplitude for two-dimensional flow: ——, $\mathscr{D}_1/\epsilon$ (3.5); —·—·—·, $\mathscr{D}_{22}/\epsilon^2$ (3.11); – – – – –, $\mathscr{D}_{23}/\epsilon^3$ (3.11).

of the kernel (2.15), (2.11) and (2.8) in (2.23) leads to an imaginary contribution from the semiresidues of $A(k_1)$ and $A(k-k_1)$ and to a principal-value integral that is real. Thus the general result for the second amplitude $a_2(k) = a_{2\mathrm{R}}(k)+\mathrm{i}a_{2\mathrm{I}}(k)$ is given explicitly for $0 < k < 1$ by

$$\begin{aligned} a_{2\mathrm{R}}(k) &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} Q_2(k,k_1)\,\mathrm{d}k_1, \\ a_{2\mathrm{I}}(k) &= k(2\pi)^{\frac{1}{2}}\left[\frac{(k-1)\,\Pi(k-1)\,\Pi(1)}{A(k-1)\,\Pi(k)}+\frac{\Pi(k+1)\,\Pi(1)}{A(k+1)\,\Pi(k)}\right], \end{aligned} \tag{3.6}$$

and a similar expression for $k > 1$. Further substitution of the kernels K_1 and K_3, which have the simple expressions

$$P_2(k,k_1) = \begin{cases} -K_1(k,k_1) = 2kk_1 & (k_1 < 0), \\ -K_3(k,k_1) = -k_1(k-k_1) & (0 < k_1 < k), \\ 0 & (k_1 > k), \end{cases} \tag{3.7}$$

and of $A(k)$ (2.10) in (3.6) yields

$$a_2(1) = \frac{(2\pi)^{\frac{1}{2}}}{\Pi(1)}\left[\mathrm{i}\Pi(1)\,\Pi(2)-\frac{1}{\pi}\int_{-\infty}^{0}\frac{\Pi(k_1)\,\Pi(1-k_1)}{1+k_1}\,\mathrm{d}k_1+\frac{1}{2\pi}\int_0^1 \Pi(k_1)\,\Pi(1-k_1)\,\mathrm{d}k_1\right]. \tag{3.8}$$

In the particular case (3.2), a_2 can be calculated in a closed analytic form by substituting (3.2) in (3.8), the result being

$$a_2(1) = \mathrm{i}a_{2\mathrm{I}}(1)+a_{2\mathrm{R}}(1) = \mathrm{i}\pi\epsilon l^2\,\mathrm{e}^{-2l}+\tfrac{1}{2}\epsilon l^2[1-2\,\mathrm{e}^{-2l}\,\mathrm{Ei}\,(2l)], \tag{3.9}$$

with Ei denoting the exponential integral.

Finally, the dimensionless wave drag based on the solution up to second order is obtained in a similar manner to the first-order drag as follows:

$$D = D_1 + D_2 = 2\pi \Pi^2(1)\,[1+a_2(1)]\,[1+a_2^*(1)]$$
$$= 2\pi \Pi^2(1)\,\{[1+a_{2\mathrm{R}}(1)]^2 + a_{2\mathrm{I}}^2(1)\}. \tag{3.10}$$

Substitution of (3.9) in (3.10) results in the following expression for the higher-order terms for the wave drag:

$$\mathscr{D}_2 = \frac{D_2}{W} = \epsilon^2 \mathscr{D}_{22} + \epsilon^3 \mathscr{D}_{23} = \epsilon^2\{\pi l^4 \,\mathrm{e}^{-2l}[1 - 2\,\mathrm{e}^{-2l}\,\mathrm{Ei}\,(2l)]\}$$
$$+ \epsilon^3\{\tfrac{1}{4}\pi l^6[1-2\,\mathrm{e}^{-2l}\,\mathrm{Ei}\,(2l)]^2 + \pi^2\,\mathrm{e}^{-6l}\}. \tag{3.11}$$

To the best of our knowledge, this, and the associated cylinder problem (Dagan & Miloh 1985), are the only ones for which closed-form second-order solutions have been found so far.

It is worthwhile to note that in (3.11) we have included terms of order ϵ^3, whereas the usual, consistent definition of the second-order drag is the term $\mathscr{D}_{22}$ solely. Nevertheless (3.11) may be interpreted as consistent in the sense that it is based on the second-order approximation of the velocity potential. Furthermore, the total drag $D_1 + D_2$ (3.11) is positive-definite for any type of pressure distribution and range of parameters. The two components $\mathscr{D}_{22}$ and $\mathscr{D}_{23}$ are represented in figure 2 as functions of the Froude number. For a fixed Fr the expansion for the drag is asymptotically uniform for $\epsilon \rightarrow 0$. This is also true for the high-Froude-number limit $l \rightarrow 0$, which yields in (3.11)

$$\frac{\mathscr{D}_2}{\mathscr{D}_1} \rightarrow \epsilon l^2[1 - 2\gamma\,\mathrm{e}^{-2l} - 2\,\mathrm{e}^{-2l}\log l + \ldots], \tag{3.12}$$

where $\gamma = 0.577\ldots$ is Euler's constant. The situation is different for the small-Froude-number limit $l \rightarrow \infty$, as the ratios between the real parts of the second- and first-order amplitude (3.9) and drag (3.11) tend to

$$\left.\begin{aligned} \frac{a_1(1)+a_{2\mathrm{R}}(1)}{a_1(1)} &= 1 + a_{2\mathrm{R}}(1) = 1 + \tfrac{1}{2}\epsilon l^2 + O(\epsilon l), \\ \frac{\mathscr{D}_1 + \mathscr{D}_2}{\mathscr{D}_1} &\rightarrow 1 + \epsilon l^2\left(1 - \frac{1}{l} - \frac{1}{2l^2} - \ldots\right) + \tfrac{1}{4}(\epsilon l^2)^2\left(1 - \frac{2}{l} - \ldots\right). \end{aligned}\right\} \tag{3.13}$$

Hence it can be seen that the small-Froude-number limit is non-uniform unless $\epsilon l^2 = o(1)$, or, in other words, no matter how small is ϵ, second-order nonlinear effects become large compared with the first-order solution when $Fr \rightarrow 0$. A graphical illustration of this statement is given in figure 3, in which the ratio $(\mathscr{D}_1 + \mathscr{D}_2)/\mathscr{D}_1$ (3.17) is represented as a function of ϵl^2. For ϵl^2 around 1.2, for instance, the ratio is already 2. Thus in the range of say $\epsilon l^2 > 0.4$ the regular perturbation expansion is no longer valid and a different approach has to be used.

The solution of the flow problem in Fourier space makes possible the interpretation of the nonlinearity of wave resistance from a new angle. Thus in the linearized solution the amplitude of the far free waves results from the selection of the wavenumber component $k = \pm 1$ of the FT of the pressure $\Pi(k)$ (3.2). In contrast, the second-order amplitude (3.6) originates from contributions of all wavenumbers of $\Pi(k)$ via the product $\Pi(k_1)\,\Pi(1-k_1)$ in the convolution integral (2.23) and the kernel Q_2 (2.15). The main result, however, is that at small Froude number the leading term stems from contributions of wave-number k_1 in the range $0 < |k_1| < 1$ solely.

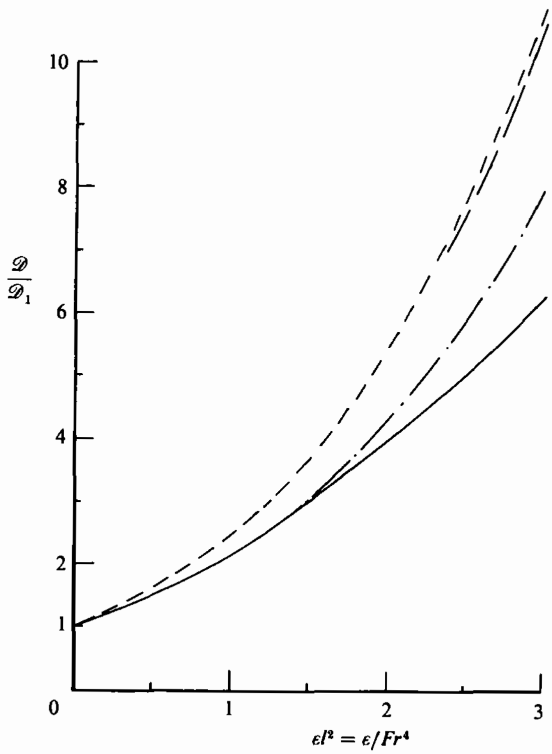

FIGURE 3. The ratio between various wave-drag nonlinear expressions and the first-order wave drag for two-dimensional flow: ——, leading-order term (3.17) for small Froude number; — — —, nonlinear generalized solution (5.9) and quasilinearized solution (6.13); —·—·—, quasilinearized solution (6.22); — — —, quasilinearized solution (6.26).

It is also advantageous for the main developments in the sequel to formalize this result by splitting the kernel Q_2 (2.15) into two parts:

$$Q_2(k, k_1) = Q_2^{\mathrm{s}}(k, k_1) + Q_2^{\mathrm{r}}(k, k_1), \tag{3.14}$$

with Q_2^{s} defined for $k > 0$ by

$$\left.\begin{aligned} Q_2^{\mathrm{s}}(k, k_1) &= \frac{k_1(k-k_1)\,\Pi(k_1)\,\Pi(k-k_1)}{(1-k_1)(1-k+k_1)\,\Pi(k)} = Q_2(k, k_1) \quad (0 < k_1 < k), \\ Q_2^{\mathrm{s}}(k, k_1) &= 0 \quad (k_1 < 0,\ k_1 > k), \end{aligned}\right\} \tag{3.15}$$

such that in (3.6) we obtain

$$a_{2\mathrm{R}}(1) = a_{2\mathrm{R}}(-1) = (2\pi)^{-\frac{1}{2}} \int_0^1 Q_2^{\mathrm{s}}(1, k_1)\,\mathrm{d}k_1 + (2\pi)^{-\frac{1}{2}} \int_{-\infty}^0 Q_2^{\mathrm{r}}(1, k_1)\,\mathrm{d}k_1 + (2\pi)^{-\frac{1}{2}} \int_1^{\infty} Q_2^{\mathrm{r}}(1, k_1)\,\mathrm{d}k_1. \tag{3.16}$$

The expression for Q_2^{s} (3.15) results from the definitions of Q_2 (2.15), P_2 (2.11), K_3 (2.8), A (2.10) and Π (3.2).

The first term in (3.16) is precisely the one that contributes to the leading order in (3.13), whereas the remaining one, associated with the kernel Q_2^{r}, is of lower order by l. Indeed, for the present example (3.2) we have in (3.16)

$$a_{2\mathrm{R}}(1) = \tfrac{1}{2}\epsilon l^2 + o(\epsilon l), \quad \frac{D_1+D_2}{D_1} = (1+\tfrac{1}{2}\epsilon l^2)^2 + o(\epsilon^2 l^2). \tag{3.17}$$

A similar result has previously been obtained by a different method (Dagan 1975) where the small-Froude-number non-uniformity has been established by examining the nature of the singularity of the complex velocity associated with the wave-making disturbance. Thus in table 1 of Dagan (1975) it is shown that the amplitude ratio a_2/a_1 is non-uniform like ϵ/Fr^2, ϵ/Fr and $\epsilon \ln Fr$ for a submerged body of length l and of a source-like, elliptical or wedge-like leading-edge shape respectively. Since in the present example the pressure distribution (3.1) is associated with the more severe singularity of a doublet type, the solution (3.17) is non-uniform like ϵ/Fr^4.

Furthermore, it was shown in the same study that these non-uniform terms originate from the Fourier transform in the complex plane of the square of the first-order complex velocity (Dagan 1975, equation (37)). Now it is easy to show that the FT of the real part of the square of an analytical function is different from zero only in the interval $0 < |k_1| < k$. Indeed, with $\mathrm{Re}\,[(\mathrm{d}f/\mathrm{d}z)^2] = (\partial\Phi/\partial x)^2 - (\partial\Phi/\partial y)^2$, where $z = x+\mathrm{i}y$ is here a complex variable, f is an analytical function and Φ denotes its real part, we have by the convolution theorem and for $y = 0$

$$\mathrm{FT}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 - \left(\frac{\partial\Phi}{\partial y}\right)^2\right] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} [-k_1(k-k_1) - |k_1|\,|k-k_1|]\,\phi(k_1)\,\phi(k-k_1)\,\mathrm{d}k_1, \tag{3.18}$$

where $\phi(k) = \mathrm{FT}\,\Phi(x,0)$. Now it is easy to ascertain that for $k > 0$ on one hand and for $k_1 < 0$, $k_1 > k$ on the other, the bracket in (3.18) is identically zero. Thus there is a close relationship between the results achieved by the two different methods, and this analogy will be discussed further in §6. The advantage of the present line of attack in comparison with the previous one resides, however, in its applicability to the more involved case of three-dimensional flows, which is illustrated in §4.

4. Illustration of regular perturbation solution (three-dimensional flow)

We consider now the more interesting problem of a three-dimensional flow, and demonstrate the general approach outline in §2 by applying it to a simple particular case. The selected pressure distribution, an immediate generalization of the two-dimensional example (3.1), is given by

$$p(x,z) = \frac{\epsilon l^4}{(l^2+x^2+z^2)^{\frac{3}{2}}}; \quad \epsilon = \frac{p'_{\max}}{\rho' g l'}; \quad W = \iint_{-\infty}^{\infty} p(x,z)\,\mathrm{d}x\,\mathrm{d}z = 2\pi\epsilon l^2. \tag{4.1}$$

where again $l = Fr^{-2}$ denotes the horizontal lengthscale, ϵ is the slenderness small parameter and W is the dimensionless weight supported by the pressure. As in the two-dimensional case (3.1), this example of an axisymmetric pressure patch has originated from the study of the motion of a submerged sphere, which is discussed in Dagan & Miloh (1985).

The FT of the pressure p (4.1) is found as follows

$$\Pi(\rho,\theta) = (2\pi)^{-1} \iint_{-\infty}^{\infty} P(x,z)\exp\left[\mathrm{i}\rho(x\cos\theta + z\sin\theta)\right]\mathrm{d}x\,\mathrm{d}z = \epsilon l^3\,\mathrm{e}^{-l\rho} \tag{4.2}$$

which is analogous to (3.2), both two- and three-dimensional $\Pi(\boldsymbol{k})$ being proportional to the weight W times an exponential factor.

The first-order wave drag can be computed in a closed form by using (2.18) with $a_1 = 1$ (2.22), which gives

$$D_1 = \pi(\epsilon l^3)^2 \int_0^{\frac{1}{2}\pi} \exp(-2l\sec^2\theta)\sec^5\theta\,\mathrm{d}\theta = \tfrac{1}{4}\pi(\epsilon l^3)^2\,\mathrm{e}^{-l}\left[K_0(l)+\left(1+\frac{1}{2l}\right)K_1(l)\right], \tag{4.3}$$

where K_n denotes the modified Bessel function of order n (see Havelock 1934).

Our main interest resides here, however, in the non-uniform behaviour at the low-Froude-number limit of the drag (4.3). The leading-order term is obtained by letting $l\to\infty$ in (4.3), and is given by

$$D_1 = \tfrac{1}{4}\pi(\epsilon l^3)^2\,\frac{2\pi}{l}\,\mathrm{e}^{-2l};\quad \mathscr{D}_1 = \frac{D_1}{W} = \tfrac{1}{8}(2\pi)^{\frac{1}{2}}\epsilon l^{\frac{5}{2}}\,\mathrm{e}^{-2l}, \tag{4.4}$$

where neglected terms are of order $\epsilon l^{\frac{3}{2}}$.

The low-Froude-number limit (4.4) could be obtained directly, without solving first exactly for D_1, with the aid of the Laplace method applied to the integral (4.3). Indeed, the argument of the exponential in (4.3) has a minimum for $\theta = \theta_0 = 0$, and by the Laplace method one has

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\theta)\,\mathrm{e}^{-lh(\theta)}\,\mathrm{d}\theta \to \left(\frac{2\pi}{lh''(\theta_0)}\right)^{\frac{1}{2}} f(\theta_0)\,\mathrm{e}^{-lh(\theta_0)} \tag{4.5}$$

for $h'(\theta_0) = 0$ and $h''(\theta_0) > 0$. By applying (4.5) to the integral (4.3) we indeed recover (4.4).

The second-order complex wave amplitude is given explicitly by (2.23). Similarly to the two-dimensional case, a_2 is made up from an imaginary term originating from the semiresidues of Q_2 and from a real, principal-value integral. Since our main concern is in the far free waves and for small Froude numbers, it can be shown in an analogous manner to the two-dimensional solution that the semiresidues of Q_2 will contribute terms $O(\mathrm{e}^{-3l})$, which are negligible compared with the principal-value term.

Proceeding with the evaluation of the real part of a_2 (2.23), we shall use the following notation:

$$\left.\begin{aligned} \boldsymbol{k} = (\rho\cos\theta, \rho\sin\theta),\quad \boldsymbol{k}_1 = (\rho_1\cos\theta_1, \rho_1\sin\theta_1)\\ \lambda^2(\theta_1) = \rho^2 - 2\rho\rho_1\cos(\theta-\theta_1)+\rho_1^2. \end{aligned}\right\} \tag{4.6}$$

Substitution of (4.6), (4.2) and (2.15) in (2.23) yields for the complex amplitude

$$\left.\begin{aligned} a_2(\rho,\theta) &= \frac{\epsilon l^3}{2\pi}\,\mathrm{e}^{l\rho}\int_0^\infty F(\rho,\theta,\rho_1)\,\mathrm{e}^{-l\rho_1}\rho_1\,\mathrm{d}\rho_1\\ F(\rho,\theta,\rho_1) &= -\int_{-\pi}^{\pi}\frac{P_2(\rho,\theta;\rho_1,\theta_1)}{A(\rho_1,\theta_1)A(\rho,\theta;\rho_1,\theta_1)}\,\mathrm{e}^{-l\lambda(\theta_1)}\,\mathrm{d}\theta_1, \end{aligned}\right\} \tag{4.7}$$

where

$$\frac{P_2(\rho,\theta;\rho_1,\theta_1)}{A(\rho_1,\theta_1)A(\rho,\theta;\rho_1\theta_1)} = \frac{\begin{aligned}\rho_1\cos\theta_1\{\rho\lambda(\theta_1)\cos\theta\,[1-\cos(\theta-\theta_1)]\\ -\tfrac{1}{2}(\rho\cos\theta-\rho_1\cos\theta_1)\,[\lambda(\theta_1)+\rho\cos(\theta-\theta_1)-\rho_1]\}\end{aligned}}{(1-\rho_1\cos^2\theta_1)\,[\lambda(\theta_1)-(\rho\cos\theta-\rho_1\cos\theta_1)^2]}. \tag{4.8}$$

Unlike the two-dimensional case, it is difficult here to obtain a closed-form expression for any l. The low-Froude-number limit can be extracted, however, by first applying the Laplace method to the function F (4.7). The leading-order term for $l \to \infty$ is obtained from $\lambda'(\theta_1) = 0$, which leads to $\theta = \theta_1$, $\lambda(\theta) = |\rho - \rho_1|$ and reduces (4.7) to

$$F(\rho, \theta, \rho_1) = \left(\frac{2\pi|\rho-\rho_1|}{\rho\rho_1 l}\right)^{\frac{1}{2}} \mathrm{e}^{-l|\rho-\rho_1|} \frac{\rho_1(\rho-\rho_1)\cos^2\theta}{(1-\rho_1\cos^2\theta)\,[1-(\rho-\rho_1)\cos^2\theta]}. \tag{4.9}$$

Finally, the contributions to the far free waves (2.17) and drag (2.18) originate from $\rho = \sec^2\theta$, which gives for the second-order wave amplitude

$$a_2(\sec^2\theta, \theta) = (2\pi)^{-\frac{1}{2}} \epsilon l^{\frac{5}{2}} \sec\theta \int_0^{\sec^2\theta} \rho_1^{\frac{1}{2}}\,(\sec^2\theta - \rho_1)^{\frac{1}{2}}\,\mathrm{d}\rho_1 = \tfrac{1}{16}(2\pi)^{\frac{1}{2}}\epsilon l^{\frac{5}{2}} \sec^5\theta, \tag{4.10}$$

and for the second-order wave drag

$$D_1 + D_2 = -\tfrac{1}{2}\pi(\epsilon l^3)^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [1 + \tfrac{1}{16}(2\pi)^{\frac{1}{2}}\epsilon l^{\frac{5}{2}} \sec^5\theta]^2\, \mathrm{e}^{-2l\sec^2\theta} \sec^5\theta\,\mathrm{d}\theta \tag{4.11}$$

On applying again the Laplace method to (4.11), precisely as in the preceding paragraph, and dividing by the first-order wave drag D_1 (4.5) yields a low-Froude-number expansion for the second-order drag

$$\frac{D_1 + D_2}{D_1} = 1 + \frac{(2\pi)^{\frac{1}{2}}}{8}\epsilon l^{\frac{5}{2}} + \frac{\pi}{128}\epsilon^2 l^5, \tag{4.12}$$

which is represented graphically in figure 5. The regular perturbation expansion is therefore non-uniform unless $\epsilon l^{\frac{5}{2}} = \epsilon/Fr^5 = o(1)$. Otherwise, no matter how small is ϵ, the second-order term becomes arbitrarily large compared with the first-order one for $Fr \to 0$. A uniform solution of the original integral equation (2.14) will be derived in §5.

Thus it is clear that, unlike the two-dimensional flow, the elementary three-dimensional far free waves propagate in the plane $y = 0$ at different angles θ. As is well known, and has been found here as well, the dominant contribution to the first-order wave drag at small Froude number originates from the transverse waves, for which $\theta = 0$. Thus, from a kinematical point of view the two- and three-dimensional cases are quite similar, since the transverse waves that trail behind the disturbance are essentially of a two-dimensional nature. The three-dimensional drag, however, suffers an additional reduction as compared with the two-dimensional one since the transverse waves represent only part of the far whole free-waves manifold. This can be easily seen by rewriting the drag (3.4) and (4.4) as follows

$$\left.\begin{aligned} \mathscr{D}_1 &= W\,\mathrm{e}^{-2l} \quad \text{(2-dimensional)}, \\ \mathscr{D}_1 &= \frac{1}{8(2\pi)^{\frac{1}{2}}}\frac{W}{l^{\frac{1}{2}}}\,\mathrm{e}^{-2l} \quad \text{(3-dimensional)}, \end{aligned}\right\} \tag{4.13}$$

after replacing ϵ by the corresponding weight function W. It is seen that the spatial effect is to reduce the three-dimensional wave drag asymptotically by a factor of $l^{-\frac{1}{2}}$ and also numerically by a factor of $(8\sqrt{(2\pi)})^{-1}$ as compared with the two-dimensional drag.

Similarly to the two-dimensional case, the dominant nonlinear contribution to the drag stems from the interaction between the various Fourier components of the pressure distribution. Furthermore, it has been shown that it is only the nonlinear interaction between the transverse waves, i.e. for $\theta_1 = \theta = 0$, that contributes to the wave drag at the leading-order term. This leads in the expression for the wave amplitude a_2 (4.10) to a summation on wavenumbers $0 < \rho_1 < 1$, similarly to the two-dimensional case (3.7). There is again a further reduction in the nonlinear effect if the two-dimensional ratio $\mathscr{D}_2/\mathscr{D}_1$ (3.13) is compared with the corresponding three-dimensional ratio (4.12). This numerical and asymptotic diminishing effect, which is related to the spread of the waves, should serve as a warning against any intuitive attempts to generalize the two-dimensional results to three-dimensional configurations.

Finally, in a similar manner to the two-dimensional flow (3.14), we can separate the kernel Q_2 in (2.23) into two parts

$$Q_2(\boldsymbol{k}, \boldsymbol{k}_1) = Q_2^{\mathrm{s}}(\rho, 0; \rho_1, \theta_1)\, \delta(\theta_1) + Q_2^{\mathrm{r}}(\rho, \theta; \rho_1, \theta_1), \tag{4.14}$$

where

$$\left.\begin{aligned} Q_2^{\mathrm{s}}(\rho, 0; \rho_1, 0) &\equiv Q_2^{\mathrm{s}}(\rho, \rho_1) \\ &= \left(\frac{2\pi}{l\rho}\right)^{\frac{1}{2}} \frac{\rho_1^{\frac{1}{2}}(\rho-\rho_1)^{\frac{3}{2}}}{(1-\rho_1)(1-\rho+\rho_1)} \frac{\Pi(\rho_1, 0)\, \Pi(\rho-\rho_1, 0)}{\Pi(\rho, 0)} \quad (0 < \rho_1 < \rho), \\ Q_2^{\mathrm{s}}(\rho, 0; \rho_1, 0) &= 0 \quad (\rho_1 > \rho), \end{aligned}\right\} \tag{4.15}$$

and $\delta(\theta_1)$ denotes the Dirac delta function in (4.14). As in the two-dimensional case, the leading term of the amplitude $a_2(\rho, 0)$, which is $O(\epsilon l^{\frac{5}{2}})$, originates in (2.14) from the integration over Q_2^{s}, whereas Q_2^{r} yields a contribution of order $\epsilon l^{\frac{3}{2}}$.

5. Derivation of small-Froude-number uniform solution

In the preceding sections we have shown that the ratios between the second- and first-order wave amplitudes a_2/a_1 become non-uniform like ϵl^2 and $\epsilon l^{\frac{5}{2}}$ for the two- or three-dimensional examples respectively. Thus the regular perturbation expansion breaks down and the naive separation of terms in the basic integral equation (2.14), which leads to (2.22)–(2.24), is not warranted.

We shall derive now a procedure that leads to a uniform perturbation expansion of $a(\boldsymbol{k})$ for $\epsilon = o(1)$ and $Fr = o(1)$ in the range $\epsilon l^2 = \epsilon\, Fr^{-4} = O(1)$ or $\epsilon l^{\frac{5}{2}} = \epsilon\, Fr^{-5} = O(1)$, for two- or three-dimensional flows respectively. In other words, we want to extend the domain of validity of the regular perturbation expansion for small Fr. The domains of uniform validity of the regular perturbation expansion and of its extension are represented schematically in figure 4.

Further extension, in the range $\epsilon l = \epsilon\, Fr^{-2} = O(1)$ for instance in two dimensions, implies retention of additional nonlinear terms beyond the leading one. Such an extension, in the third region of figure 4, is not considered here.

Towards this goal, we rewrite the basic integral equation (2.14) as

$$a(k) - (2\pi)^{-\frac{1}{2}} \int_0^k Q_2^{\mathrm{s}}(k, k_1)\, a(k_1)\, a(k-k_1)\, \mathrm{d}k_1 = 1 + (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} Q_2^{\mathrm{r}}(k, k_1)\, a(k_1)\, a(k-k_1)\, \mathrm{d}k_1 \tag{5.1}$$

for the two-dimensional flow with Q_2^{s} given by (3.15), and

$$a(\rho, 0) - (2\pi)^{-1} \int_0^{\rho} Q_2^{\mathrm{s}}(\rho, \rho_1)\, a(\rho_1, 0)\, a(\rho-\rho_1, 0)\, \rho_1\, \mathrm{d}\rho_1 = 1 + (2\pi)^{-1} \int\!\!\int_{-\infty}^{\infty} Q_2^{\mathrm{r}}(\boldsymbol{k}, \boldsymbol{k}_1)\, a(\boldsymbol{k}_1)\, a(\boldsymbol{k}-\boldsymbol{k}_1)\, \mathrm{d}\boldsymbol{k}_1 \tag{5.2}$$

for the three-dimensional flow with Q_2^{s} given by (4.15).

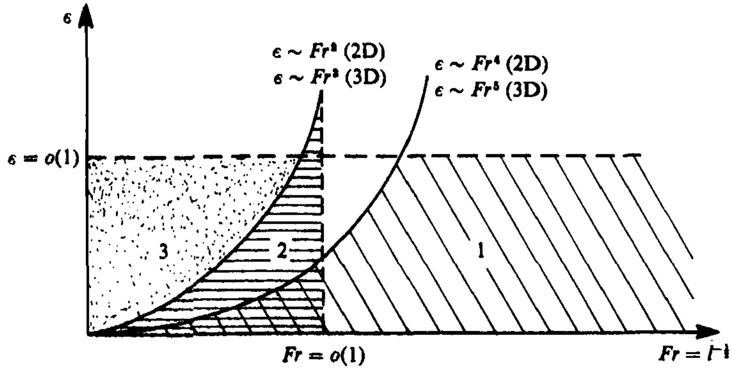

FIGURE 4. A schematic representation of the domains of uniform validity of (1) the regular perturbation expansion $\epsilon = o(1)$, $Fr = O(1)$; (2) the generalized solution $\epsilon = o(1)$, $Fr = o(1)$, ϵ/Fr^2 or $\epsilon/Fr^3 = o(1)$ and (3) $\epsilon = o(1)$, arbitrary Froude number.

Under a regular perturbation expansion for $\epsilon = o(1)$ and in the ranges of Fr mentioned before, the two terms of the left-hand sides of (5.1) and (5.2) become of the same order. They have to be kept in the same equation, therefore, to ensure a uniform expansion. In contrast, the last term of the right-hand side of (5.1) and (5.2) is asymptotic to $a(\boldsymbol{k})$ under the same limits and for $|\boldsymbol{k}| = 1$.

Our basic idea is to expand the complex wave amplitude $a(\boldsymbol{k})$ in a generalized perturbation expansion

$$a^{g}(\boldsymbol{k};\epsilon,l) = a_1^{g}(\boldsymbol{k};\epsilon,l)+a_2^{g}(\boldsymbol{k};\epsilon,l)+\dots \tag{5.3}$$

such that after substitution in (5.1) or (5.2), a_1^g satisfies the following integral equation:

$$a_1^{g}(\boldsymbol{k})-(2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|} Q_2^{g}(\boldsymbol{k},\boldsymbol{k}_1)\,a_1^{g}(\boldsymbol{k}_1)\,a_1^{g}(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 = 1, \tag{5.4}$$

with $m = 1$ or 2 for two- or three-dimensional flows respectively. The next term in the generalized expansion, a_2^g, satisfies a similar integral equation, which is obtained by grouping terms of the next order, i.e.

$$\begin{aligned} a_2^{g}(\boldsymbol{k})-2(2\pi)^{-\frac{1}{2}m}&\int_0^{|\boldsymbol{k}|} Q_2^{g}(\boldsymbol{k},\boldsymbol{k}_1)\,a_1^{g}(\boldsymbol{k}_1)\,a_2^{g}(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 \\ &= (2\pi)^{-\frac{1}{2}m}\int_{-\infty}^{\infty} Q_2^{r}(\boldsymbol{k},\boldsymbol{k}_1)\,a_1^{g}(\boldsymbol{k}_1)\,a_1^{g}(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1, \end{aligned} \tag{5.5}$$

and so forth.

The analysis of the previous sections implies indeed that (5.3) is a uniform expansion for the values of $\boldsymbol{k}$ that give the far free waves. Furthermore, an expansion of a_1^g for $\epsilon = o(1)$ and $Fr = O(1)$ recovers the first- and second-order leading terms of a_1 and a_2. The domains of uniformity of the regular and generalized expansions are illustrated schematically in figure 4.

It should be emphasized that (5.4) is a Volterra nonlinear integral equation, which is considerably simpler to handle numerically than the original integral equation (2.14). This will become apparent as we proceed with the derivation of a_1^g for the selected pressure distribution.

Indeed, the integral equation (5.4) can be rewritten for two-dimensional flow with

Q_2 given by (3.15) and Π by (3.2) as

$$a_1^{\mathrm{g}}(k) - \tfrac{1}{2}\epsilon l^2 \int_0^k \frac{k_1(k-k_1)}{(1-k_1)(1-k+k_1)} a_1^{\mathrm{g}}(k_1)\, a_1^{\mathrm{g}}(k-k_1)\, \mathrm{d}k_1 = 1. \tag{5.6}$$

This Volterra nonlinear integral equation can be easily solved in an explicit form by a standard numerical procedure by discretizing the interval $0 < k_1 < k$. Thus, with k replaced by $k_n = n/N$ $(n = 0, 1, 2, \ldots, N)$, $a_1^{\mathrm{g}}(k_n)$ can be determined in terms of $a_1^{\mathrm{g}}(k_{n-1})$ by a single algebraic operation from (5.6) as follows:

$$\alpha a_1^{\mathrm{g}}(k_n) = 1 + \tfrac{1}{2}\epsilon l^2 \sum_{m=0}^{n-1} \frac{k_m(k_n-k_m)}{(1-k_m)(1-k_n+k_m)}\, a_1^{\mathrm{g}}(k_m)\, a_1^{\mathrm{g}}(k_n-k_m)$$
$$\left(k_m = \frac{m}{N}, \quad k_n = \frac{n}{N}, \quad m, n = 1, 2, \ldots, N\right), \tag{5.7}$$

where $\alpha = 1$ for $n = N$ and $\alpha = 1 - \frac{1}{2}\epsilon l^2/N$ for $n = N$, since the kernel in (5.6) is unity for $k = k_1 = 1$.

It should also be emphasized that in (5.7) the summation is truncated at $m = n-1$ because the term $k_n - k_m$ vanishes for $m = n = N$.

Similarly, by substituting Q_2^{s} (4.15) and Π (4.2) into (5.4) for three-dimensional flow, we get

$$a_1^{\mathrm{g}}(\rho, 0) - \frac{\epsilon l^{\frac{5}{2}}}{(2\pi)^{\frac{1}{2}}} \int_0^{\rho} \frac{\rho_1^{\frac{3}{2}}(\rho-\rho_1)^{\frac{3}{2}}}{\rho^{\frac{1}{2}}(1-\rho_1)(1-\rho+\rho_1)}\, a_1^{\mathrm{g}}(\rho_1, 0)\, a_1^{\mathrm{g}}(\rho-\rho_1)\, \mathrm{d}\rho_1 = 1. \tag{5.8}$$

This equation can be easily solved again by the same numerical procedure which led to (5.7), with $\alpha = 1$ for all n, since, unlike (5.6), the three-dimensional kernel (5.8) vanishes for $\rho_1 = \rho = 1$.

It is seen that the unknown coefficients of the far free-wave amplitudes, $a_1^{\mathrm{g}}(1)$ and $a_1^{\mathrm{g}}(1, 0)$, depend only on the parameters $\epsilon l^2 = \epsilon/Fr^4$ and $\epsilon l^{\frac{5}{2}} = \epsilon/Fr^5$ respectively.

Once these coefficients are determined by solving (5.7) or (5.8), the generalized wave drag is given by

$$\left.\begin{aligned} \frac{\mathscr{D}_1^{\mathrm{g}}}{\mathscr{D}_1} &= [a_1^{\mathrm{g}}(1)]^2 \quad \text{(2-dimensional)}, \\ \frac{\mathscr{D}_1^{\mathrm{g}}}{\mathscr{D}_1} &= [a_1^{\mathrm{g}}(1,0)]^2 \quad \text{(3-dimensional)}, \end{aligned}\right\} \tag{5.9}$$

by virtue of (2.20) or (3.4) for two-dimensional flow or (4.4) for three-dimensional flow.

The results of the numerical solution for the generalized wave drag (5.9) are represented in figure 3 (two-dimensional flow) and in figure 5 (3-dimensional flow) as functions of ϵl^2 and $\epsilon l^{\frac{5}{2}}$ respectively. It is seen that the nonlinear drag grows faster with these arguments than the second-order perturbation solutions (3.17) and (4.12). Furthermore, the latter is seen to depart significantly from the generalized solution for $\epsilon l^2 > 0.8$ and $\epsilon l^{\frac{5}{2}} > 3$ respectively. Thus, in these examples the non-uniformity of the regular perturbation expansion is more severe in two- than in three-dimensional flow.

This completes the derivation of the first-order generalized solution which is uniformly valid at second-order for the ordering

$$\epsilon = o(1), \quad Fr = o(1), \quad \epsilon l^2 = \epsilon/Fr^4 = O(1)$$

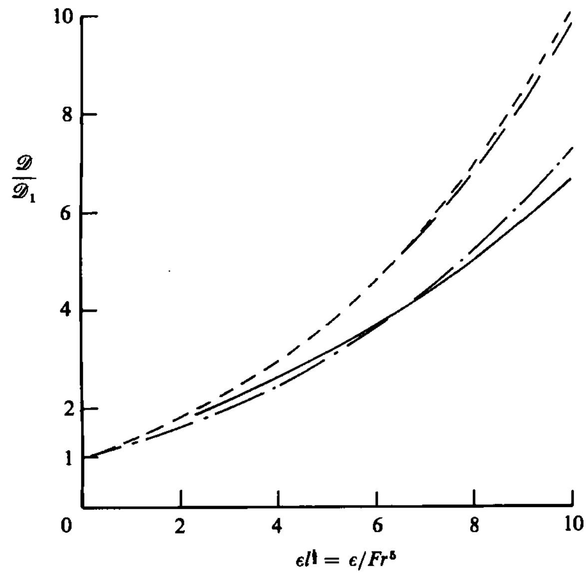

FIGURE 5. The ratio between various wave-drag nonlinear expressions and the first-order wave drag for three-dimensional flow for small Froude number: ——, leading-order term (4.12) of regular perturbation expansion; ---, generalized solution (5.9) and quasilinearized solution (6.13); —·—·—, quasilinearized solution (6.22); ———, quasilinearized solution (6.26).

in the two-dimensional example and

$$\epsilon = o(1), \quad Fr = o(1), \quad \epsilon l^{\frac{5}{2}} = \epsilon/Fr^5 = O(1)$$

in the three-dimensional example.

In Appendix B we evaluate the highest-order contributions for $Fr \rightarrow 0$ in the third-order perturbation term (2.24). These contributions are found, in the present examples, to be rather small compared with the second-order terms.

6. Quasilinearized generalized solutions and comparison with existing approaches

As we have already indicated in § 1, the existing approximate methods to compute wave resistance in three-dimensional flows for small Froude number are based on the quasilinearization of the free-surface conditions. Furthermore, the approximate equations are formulated in physical rather than in Fourier space. In the present section we are going to compare the generalized solution of § 5 with a few existing quasilinearized approximations.

6.1. *Formulation of the free-surface conditions for the generalized solution in physical space*

The generalized first-order amplitude function $a_1^g(\boldsymbol{k})$ satisfies the Volterra integral equation (5.4). We wish to find out what are the free-surface boundary conditions in physical space which would have led to the same integral equation in Fourier space. It is easy to ascertain that the kernel Q_2^s defined by (3.14) and (3.15) for the two-dimensional example originates from the kernel K_3 (2.8) via P_2 (2.11). Indeed, in the interval $0 < k_1 < k$, for $k > 0$, in which Q_s is different from zero, we have in (2.8) $K_1 \equiv 0$ and $K_3 = k_1(k-k_1)$, whereas, for $k_1 > k, K_3 \equiv 0$. Thus the FT of the free-surface conditions replacing (2.6) and (2.7) at second order and leading to the generalized solution are

$$\mathrm{i}k\psi(k)+\zeta(k) = -\Pi(k)+(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} K_3(k,k_1)\,\psi(k_1)\,\psi(k-k_1)\,\mathrm{d}k_1, \tag{6.1}$$

$$-\mathrm{i}k\,\zeta(k)+|k|\,\psi(k) = 0. \tag{6.2}$$

By inverting (6.1) and (6.2), it is easy to ascertain that they could be obtained from the FT of the following boundary conditions in the physical plane, replacing (2.2):

$$\frac{\partial \Phi}{\partial x}-\eta-\frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2 = p(x) \quad (y=0), \tag{6.3}$$

$$\frac{\partial \eta}{\partial x}+\frac{\partial \Phi}{\partial y} = 0 \quad (y=0). \tag{6.4}$$

It should be emphasized that in the original equations (2.6) and (2.7) ψ stands for the FT of Ψ, the potential Φ on $y=\eta$, whereas (6.3) and (6.4) are written down for Φ on $y=0$. Furthermore, it can be seen that in the quadratic term in (6.3) the sign of $\frac{1}{2}(\partial\Phi/\partial y)^2$ is different from that of the Bernoullian term of (2.2). This change can be traced back to (2.3), and results from the relationship

$$\partial\Psi/\partial x = \partial\Phi/\partial x+(\partial\Phi/\partial y)\,\partial\eta/\partial x.$$

Elimination of η from (6.3) and (6.4) yields the following boundary condition for the potential:

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial \Phi}{\partial y}+\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial \Phi}{\partial y}\right)^2-\left(\frac{\partial \Phi}{\partial x}\right)^2\right] = \frac{\partial p(x)}{\partial x} \quad (y=0). \tag{6.5}$$

An interesting feature of this boundary condition is that it can be rewritten in terms of analytical functions. Indeed, with $w(z)$ denoting the complex velocity, such that $\partial\Phi/\partial x = \operatorname{Re} w$, (6.5) may be rewritten as

$$\operatorname{Re}\left[\frac{\partial w}{\partial z}+\mathrm{i}w-\frac{1}{2}\frac{\mathrm{d}(w^2)}{\mathrm{d}z}\right] = \frac{\partial p}{\partial x} \quad (y=0). \tag{6.6}$$

This interpretation of the free-surface boundary condition connects the present analysis with the one carried out previously (Dagan 1975), as mentioned already in §2. It is worthwhile to mention here again that the two formulations, i.e. the one in which the quadratic term in (6.6) is expressed by w^2 or the resulting Volterra integral equation (5.6) in Fourier space, are intimately related. This point deserves further investigation in relation with other free-surface two-dimensional flows.

Turning now to the example of three-dimensional flow, it is easy to ascertain by the same procedure of tracing back that the generalized integral equation (5.8) also stems from the same simplified boundary condition (6.5). As a matter of fact, (6.5) leads to a more complicated integral equation than (5.8), which is obtained only asymptotically for $Fr \to 0$. This point can be understood if we rewrite the kernel associated with the convolution originating from the nonlinear terms in (6.5) as

$$\begin{aligned} K_3'(\rho,\theta;\rho_1,\theta_1) &= \tfrac{1}{2}[|\boldsymbol{k}_1|\,|\boldsymbol{k}-\boldsymbol{k}_1|+\boldsymbol{k}_1\cdot(\boldsymbol{k}-\boldsymbol{k}_1)] \\ &= \tfrac{1}{2}\{\rho_1[\rho^2-2\rho\rho_1\cos(\theta-\theta_1)+\rho_1^2]^{\frac{1}{2}}+\rho\rho_1\cos\theta_1\cos\theta-\rho_1^2\cos^2\theta_1\}. \end{aligned} \tag{6.7}$$

Only under the asymptotic analysis of §4, which leads to $\theta = \theta_1 = 0$, does K_3' reduce to the simple expression $K_3' = \rho_1(\rho - \rho_1)$ for $\rho_1 < \rho$ and $K_3' = 0$ for $\rho_1 > \rho$, which in turn yields the Volterra integral equation (5.8). By the same token we can regard the generalized solution as one originating asymptotically from the more involved boundary condition

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial y} + \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial \Phi}{\partial y}\right)^2 - \left(\frac{\partial \Phi}{\partial x}\right)^2 - \left(\frac{\partial \Phi}{\partial z}\right)^2\right] = \frac{\partial p(x,z)}{\partial x} \quad (y = 0). \tag{6.8}$$

Thus an important conclusion for the present example is that by operating in Fourier space we are able to further simplify the three-dimensional flow problem beyond the similar result in physical space. In other words, (6.5) or (6.8) contain terms whose FT drop out in the small-Froude-number limit which leads ultimately to (5.8). This finding suggests that it is worthwhile to pursue the line of attack followed here for further investigation of three-dimensional flows. Still, a comparison with existing approaches will be carried out by using (6.5) and (6.8) as a reference.

6.2. *Solution by full quasilinearization*

The quasilinearization of the free-surface condition is achieved by substituting

$$\Phi^{\mathrm{ql}} = \Phi_1 + \varphi \tag{6.9}$$

in (6.5) or (6.8) and by neglecting terms $O(\varphi^2)$. This procedure yields

$$\frac{\partial^2 \Phi^{\mathrm{ql}}}{\partial x^2} + \frac{\partial \Phi^{\mathrm{ql}}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi^{\mathrm{ql}}}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Phi_1}{\partial y}\frac{\partial \Phi^{\mathrm{ql}}}{\partial y}\right)$$

$$= \frac{\partial p(x)}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial \Phi_1}{\partial y}\right)^2 - \left(\frac{\partial \Phi_1}{\partial x}\right)^2\right] \quad (y = 0) \tag{6.10}$$

in lieu of (6.5). The integral equation for the generalized solution based on (6.10) is achieved by applying the FT to (6.10) and also an additional small Froude-number limit for three-dimensional flow. It is simpler, however, to carry out the quasilinearization directly in the integral equations (5.4), (5.6) and (5.8) by substituting

$$a^{\mathrm{ql}}(\boldsymbol{k}) = 1 + \delta(\boldsymbol{k}) \tag{6.11}$$

and by subsequently neglecting terms $O(\delta^2)$. The result is

$$a^{\mathrm{ql}}(\boldsymbol{k}) - 2(2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|} Q_2^{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k}_1)\, a^{\mathrm{ql}}(\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 = 1 - (2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|} Q_2^{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 \tag{6.12}$$

replacing (5.4), and similar linear Volterra integral equations replacing (5.6) or (5.8). These equations have been solved by the numerical procedure of §5 and the results have been substituted in the expression of the drag (5.9):

$$\left.\begin{aligned} \frac{\mathscr{D}^{\mathrm{ql}}}{\mathscr{D}_1} &= [a^{\mathrm{ql}}(1)]^2 \quad \text{(2-dimensional)},\\ \frac{\mathscr{D}^{\mathrm{ql}}}{\mathscr{D}_1} &= [a^{\mathrm{ql}}(1,0)]^2 \quad \text{(3-dimensional)}.\end{aligned}\right\} \tag{6.13}$$

Surprisingly enough, the values were identical (to the forth digit), in the range of values of ϵl^2 and $\epsilon l^{\frac{5}{2}}$ of figure 3 and 5 respectively, with the computed nonlinear drag (5.9). We could not find a general explanation for this result, and further investigation should show whether it is fortuitous or not.

6.3. *Quasilinearization by a 'rigid-wall' approximation (Dawson approach)*

Another quasilinearized approximation, which has been employed in the past, is obtained by substituting

$$\Phi^{\mathrm{D}} = \Phi^{\mathrm{rw}} + \varphi \tag{6.14}$$

in (6.5) and (6.8) and neglecting terms $O(\varphi^2)$. In the case of flow past submerged or floating bodies Φ^{rw} is the rigid-wall solution satisfying the condition $\partial\Phi^{\mathrm{rw}}/\partial y = 0$ on $y = 0$ (obviously Φ^{rw} is singular within the flow domain $y < 0$). The derivation of various small-Froude-number approximations of this type are reviewed in detail by Tulin (1978). The development of this type of approximation for flow generated by a pressure distribution requires a few preparatory steps (see discussion in Doctors & Dagan 1980). The simplest way to arrive at the appropriate 'naive' small-Froude-number approximation, similar to the 'rigid-wall' approximation, is to analyse the FT of the first-order solution (2.26), which yields for $\psi_1(\boldsymbol{k})$

$$\psi_1(\boldsymbol{k}) = \frac{\mathrm{i}k_x}{|\boldsymbol{k}|}\zeta_1(\boldsymbol{k}) = -\frac{ik_x}{|\boldsymbol{k}|}\frac{\Pi(\boldsymbol{k})}{A(\boldsymbol{k})}. \tag{6.15}$$

We now rewrite (6.15) by using variables made dimensionless with respect to U' and l', i.e. $\boldsymbol{K} = \boldsymbol{k}l = \boldsymbol{k}/Fr^2$, and obtain by (2.10)

$$\psi_1(\boldsymbol{K}) = -\frac{\mathrm{i}K_x|\boldsymbol{K}|\Pi(\boldsymbol{K})}{|\boldsymbol{K}|^2 - Fr^2 K_x}. \tag{6.16}$$

The 'naive' small-Froude-number approximation is obtained by expanding the denominator in (6.16) for fixed $\boldsymbol{K}$ and $Fr^2 \to 0$, the first approximation, corresponding to the 'rigid-wall' solution for a body, being

$$\psi^{\mathrm{rw}}(\boldsymbol{k}) = -\frac{\mathrm{i}k_x \Pi(\boldsymbol{k})}{|\boldsymbol{k}|}. \tag{6.17}$$

It can immediately be seen by inversion in (6.17) that Φ^{rw} satisfies the free-surface condition

$$\frac{\partial\Phi^{\mathrm{rw}}}{\partial y} = \frac{\partial p}{\partial x} \quad (y = 0), \tag{6.18}$$

which is precisely the result obtained by Doctors & Dagan (1980). We carry out now the quasilinearization of (6.15) by substituting $\Phi^{\mathrm{D}} = \Phi^{\mathrm{rw}} + \varphi$ for Φ, the result being precisely (6.10) with Φ_1 replaced by Φ^{rw}, i.e.

$$\frac{\partial^2\Phi^{\mathrm{D}}}{\partial x^2} + \frac{\partial\Phi^{\mathrm{D}}}{\partial y} - \frac{\partial}{\partial x}\left[\frac{\partial\Phi^{\mathrm{rw}}}{\partial x}\frac{\partial\Phi^{\mathrm{D}}}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial\Phi^{\mathrm{rw}}}{\partial y}\frac{\partial\Phi^{\mathrm{D}}}{\partial y}\right] = \frac{\partial p}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\Phi^{\mathrm{rw}}}{\partial y}\right)^2 - \left(\frac{\partial\Phi^{\mathrm{rw}}}{\partial x}\right)^2\right] \quad (y = 0). \tag{6.19}$$

Equation (6.19) has a structure similar, up to second-order terms, to the approximation suggested by Dawson (1977) for a submerged body or a ship (for discussion see Tulin 1982).

Now, it is easy to solve the Volterra linear integral equation based on (6.19). Towards this aim we have to quasilinearize (5.4) by substituting

$$a^{\mathrm{D}}(\boldsymbol{k}) = A(\boldsymbol{k}) + \delta(\boldsymbol{k}) \tag{6.20}$$

and neglecting terms $O(\delta^2)$. The resulting equation is

$$a^{\mathrm{D}}(\boldsymbol{k})-2(2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|}Q_2^{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k}_1)A(\boldsymbol{k}-\boldsymbol{k}_1)\,a^{\mathrm{D}}(\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 = 1-(2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|}Q_2^{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k}_1)A(\boldsymbol{k}-\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1. \quad (6.21)$$

This equation has been solved numerically for both two-and three-dimensional examples, and the corresponding wave drags

$$\left.\begin{aligned}\frac{\mathscr{D}^{\mathrm{D}}}{\mathscr{D}_1}&=[a^{\mathrm{D}}(1)]^2 \quad \text{(2-dimensional)},\\ \frac{\mathscr{D}^{\mathrm{D}}}{\mathscr{D}_1}&=[a^{\mathrm{D}}(1,0)]^2 \quad \text{(3-dimensional)},\end{aligned}\right\} \quad (6.22)$$

are depicted in figures 3 and 5 respectively. It can be seen that the quasilinearization procedure by Φ^{rw} leads to a significant underestimation of the wave drag as compared with the generalized solution (5.9).

This discrepancy can be explained by analysing the approximation leading from ψ_1 (6.15) to ψ^{rw} (6.17). It can be seen that the 'naive' approximation ψ^{rw} is valid for wavenumbers $|\boldsymbol{k}| \ll 1$, far from the pole of $[A(\boldsymbol{k})]^{-1}$. But in the Volterra integral equation (6.12) the range of integration of $\boldsymbol{k}_1$ extends to $|\boldsymbol{k}_1| = 1$, making the replacement of Φ_1 by Φ^{rw} in (6.10) somewhat questionable.

6.4. *Quasilinearization by a 'rigid-wall' approximation* (*Inui–Kajitani approach*)

Inui & Kajitani (1977) have suggested a further approximation in the free-surface condition related to wave-making by a body, namely to neglect the nonlinear term in the right-hand side of (6.19). Thus the corresponding approximation here is

$$\frac{\partial^2\Phi^{\mathrm{IK}}}{\partial x^2}+\frac{\partial\Phi^{\mathrm{IK}}}{\partial y}-\frac{\partial}{\partial x}\left(\frac{\partial\Phi^{\mathrm{rw}}}{\partial x}\frac{\partial\Phi^{\mathrm{IK}}}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\Phi^{\mathrm{rw}}}{\partial y}\frac{\partial\Phi^{\mathrm{IK}}}{\partial y}\right)=\frac{\partial p}{\partial x} \quad (y=0), \quad (6.23)$$

and the related integral equation replacing (6.21) is given by

$$a^{\mathrm{IK}}(\boldsymbol{k})-2(2\pi)^{-\frac{1}{2}m}\int_0^{|\boldsymbol{k}|}Q_2^{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k}_1)A(\boldsymbol{k}-\boldsymbol{k}_1)\,a^{\mathrm{IK}}(\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1 = 1. \quad (6.24)$$

It is worthwhile to mention that in the range $0 < |\boldsymbol{k}_1| < |\boldsymbol{k}|$ the contributions from the two nonlinear terms in $(\partial\phi/\partial x)^2$ and $-(\partial\phi/\partial y)^2$ (6.5) are the same, which is easily seen from the expression of K_3 (2.8). Hence at the same order we could rewrite (6.23) as

$$\frac{\partial}{\partial x}\left[\left(1-2\frac{\partial\Phi^{\mathrm{rw}}}{\partial x}\right)\frac{\partial\Phi^{\mathrm{IK}}}{\partial x}\right]+\frac{\partial\Phi^{\mathrm{IK}}}{\partial y}=\frac{\partial p}{\partial x} \quad (y=0), \quad (6.25)$$

which indeed in similar to, though not identical with, the Inui & Kajitani (1977) approximation (for a similar discussion see also Tulin 1982).

The wave drag based on this type of approximation,

$$\left.\begin{aligned}\frac{\mathscr{D}^{\mathrm{IK}}}{\mathscr{D}_1}&=[a^{\mathrm{IK}}(1)]^2 \quad \text{(2-dimensional)},\\ \frac{\mathscr{D}^{\mathrm{IK}}}{\mathscr{D}_1}&=[a^{\mathrm{IK}}(1,0)]^2 \quad \text{(3-dimensional)},\end{aligned}\right\} \quad (6.26)$$

is represented as well in figures 3 and 5 after solving (6.24). Unexpectedly, $\mathscr{D}^{\mathrm{IK}}$ is very close to the nonlinear generalized solution $\mathscr{D}_1^{\mathrm{g}}$ (5.9) or to the full quasilinearized approximation (6.13). In view of the observation at the end of the previous paragraph, this agreement can be explained only by mutual cancellation of effects of opposite signs. It is nevertheless worthwhile to mention that good agreement has been found between this type of approximation and second-order perturbation solution for a different two-dimensional pressure distribution by Doctors & Dagan (1980).

7. Summary and conclusions

The problem of wave generation and wave resistance of steadily moving disturbances has been formulated in Fourier space by a procedure similar to the one leading to the Zakharov equation. The resulting nonlinear integral equation is first solved by the usual regular perturbation expansion. For the particular examples of travelling pressure distributions considered here it is found that the expansion of the wave amplitude and wave resistance is non-uniform for small Froude numbers, in agreement with results obtained previously by other methods for two-dimensional flows. The expansion of the velocity potential in Fourier space also permits one to single out the nonlinear term responsible for the leading contribution at small Froude number.

By maintaining these terms in the first-order approximation of the free-surface condition, a generalized, uniformly valid, solution is obtained. It is found to satisfy a Volterra nonlinear integral equation, which lends itself to a rather simple numerical solution. The generalized wave drag is shown to be significantly larger than the one derived by a regular perturbation expansion, especially for the two-dimensional flow example.

While the results for two-dimensional flow are merely a confirmation of those of previous studies, the analysis of three-dimensional flow provides new insight into the complicated nonlinear wave-making problem. Indeed, nonlinear three-dimensional wave-resistance analysis has been confined in the past to involved numerical solutions of only a few examples. Besides the elucidation of the origin of the leading nonlinear term at small Froude number, the generalized solution presented here gives an opportunity to examine a few existing approximate approaches. The main result of the analysis, for the particular case at hand, is the confirmation of the belief that nonlinear free-surface effects can be accounted for by a quasilinearization of the free-surface condition. In the present example it was found that very good agreement with the generalized nonlinear solution may be obtained by quasilinearization with the aid of the first-order solution or by a procedure similar to that employed by Inui & Kajitani (1977). It is worthwhile to mention here that the quasilinearization procedure can be also interpreted as wave generation and propagation on a non-uniform current, since it results in linear free-surface boundary conditions with variable coefficients.

In view of the simplification achieved by solving the problem in Fourier space, it is suggested that analysis of additional examples of three-dimensional flows by the same methodology is a worthwhile venture.

We are grateful to M. P. Tulin, whose comments have greatly contributed to the revision and expansion of the first version of the present study.

Appendix A. Derivation of (2.20)

The two-dimensional wave resistance of a disturbing body moving with velocity c below and parallel to an undisturbed free surface is related to the mean energy per unit area of the free surface $\bar{E}$ by (Lamb 1935, p. 415)

$$D = \left(\frac{c_g}{c} - 1\right)\bar{E}, \tag{A 1}$$

where c_g is the group velocity. The dimensionless mean energy is

$$\bar{E} = \frac{1}{4\pi}\int_0^{2\pi} \eta^2 \,\mathrm{d}x - \frac{1}{4\pi}\int_0^{2\pi} \phi\frac{\partial \eta}{\partial x}\,\mathrm{d}x. \tag{A 2}$$

where the second term in the right-hand side of (A 2), representing the kinetic energy of the fluid, is obtained by applying the Gauss transformation to the kinematic free-surface boundary conditions (2.2). The periodic properties of the velocity potential ϕ and the free-surface elevation η over the interval $0 < x < 2\pi$, together with the dynamic free-surface boundary conditions of zero pressure (2.2), yield

$$\bar{E} = \frac{1}{2\pi}\int_0^{2\pi} \eta^2\,\mathrm{d}x + \frac{1}{8\pi}\int_0^{2\pi} \eta\left[\eta^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]\mathrm{d}x + O(\epsilon^4), \tag{A 3}$$

where ϵ here denotes the wave steepness (wave amplitude times wavenumber). Hence, to third order in ϵ, the total mean energy depends only on the free-surface elevation. Substituting the Stokes expansion (2.19) with $\eta^{(m)} = O(\epsilon^m)$ in (A 3) shows that the only surviving terms of $O(\epsilon^3)$ result from the first harmonic of the first integral in (A 3), whereas the higher harmonics and the second integral contribute only terms $O(\epsilon^4)$. The mean total energy to third order is therefore

$$\bar{E} = 2\eta^{(1)}\eta^{(1)*} + O(\epsilon^4). \tag{A 4}$$

Finally, since the ratio between the group velocity and the wave celerity for finite-amplitude waves in deep water is $c_g/c = \frac{1}{2} + O(\epsilon^2)$ (Lamb 1935, p. 417), equation (2.20) for the wave resistance follows immediately from (A 1) and (A 4).

Appendix B. Analysis of third-order effects for small Froude numbers

An explicit expression of the third-order approximation is given by (2.24), and the question is whether this term also becomes non-uniform for $\epsilon l^2 = O(1)$ and $\epsilon l^{\frac{5}{2}} = O(1)$ for the two- and three-dimensional pressure distributions respectively. Although the computations are tedious, closed-form expressions can be obtained for $a_3(1)$ (2.24) at the leading-order term for $l \to \infty$, i.e. $Fr \to 0$.

Starting with the two-dimensional example, it can be shown that, in the integration process in the (k_1, k_2)-plane that gives a_3 in (2.24), the higher-order contribution originates from the portion of the plane for which $|k-k_1|+|k_1-k_2|+|k_2| = k$ for $k > 0$. It may be recalled that a similar condition singled out the second-order contribution in §3. It is easily found that the corresponding area in the plane is the triangle $0 < k_2 < k_1$, $0 < k_1 < k$; thus (2.24) yields

$$a_3(k) = (2\pi)^{-1}\int_0^k \mathrm{d}k_1 \int_0^{k_1} Q_3'(k, k_1, k_2)\,\mathrm{d}k_2 + O(\epsilon^3 l^5), \tag{B 1}$$

where the two-dimensional kernel $Q_3'(k, k_1, k_2)$, evaluated over the triangle $0 < k_1 < k$, $0 < k_2 < k_1$, is obtained by substituting (2.8), (2.11), (2.12) and (2.15) into (2.25), which gives

$$Q_3'(1,k_1,k_2) = \frac{\pi}{2}(\epsilon l^2)^2\left[-\frac{k_2[2k_1-k_2-1+2(1-k_1)(k_1-k_2)]}{2k_1(1-k_2)(1-k_1+k_2)}+\frac{2k_2(k_1-k_2)}{(1-k_2)(1-k_1+k_2)}\right]. \tag{B 2}$$

Performing the double integration in (B 1) for $k = 1$, with Q_3' given in (B 2), gives, for two-dimensional flow,

$$a_3(1) = (2\pi)^{-1}\int_0^1\int_0^{k_1} Q_3'(1,k_1,k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2 = \frac{1}{8}\left(\frac{7\pi^2}{6}-11\right)(\epsilon l^2)^2. \tag{B 3}$$

It should be emphasized that $a_3(1)$ is the only term that contributes at leading order to the wave drag. There is a third-order correction, which results in a shift of the velocity or the basic wavenumber well known in the theory of Stokes waves (see e.g. Wehausen & Laitone 1960). This effect upon wave resistance has been analysed in detail by Doctors & Dagan (1980), and it is negligible for small *Fr*.

Examination of the two-dimensional third-order term a_3 (B 3) shows that this term is non-uniform under the limit $\epsilon l^2 = O(1)$, and strictly speaking it has to be incorporated in the generalized integral equation (5.4). It turns out, however, that there is a considerable numerical reduction of this contribution as compared with a_2 (3.17) and a_1 (2.22). Indeed, we have the following ratios from these equations:

$$\frac{a_2(1)}{a_1(1)} = \tfrac{1}{2}\epsilon l^2;\quad \frac{a_3(1)}{a_2(1)} = 0.1286\epsilon l^2,\quad \frac{a_3(1)}{a_1(1)+a_2(1)} = \frac{0.1286\,(\epsilon l^2)^2}{1+0.5\epsilon l^2}. \tag{B 4}$$

Thus the third-order correction to the wave drag (2.20) is quite small. Furthermore, a derivation of the generalized solution including the third-order term by quasi-linearization along the lines of §5 led to the same conclusion: for the selected pressure distribution the third-order correction at small Froude number is small compared with the second-order one. This is in contrast with the conventional Zakharov integral equation, in which the nonlinear wave interaction is governed by the third-order effect.

A similar analysis was also carried out for the three-dimensional distribution (4.1) by employing the Laplace method. The third-order term (2.24) is now

$$a_3(\rho,\theta) = (2\pi)^{-2}\int\int_{-\infty}^{\infty}\int\int_{-\pi}^{\pi} Q_3''(\rho,\theta;\rho_1,\theta_1;\rho_2,\theta_2)\,\mathrm{e}^{-l(f-\rho)}\,\rho_1\rho_2\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2, \tag{B 5}$$

where

$$f(\rho,\theta;\rho_1,\theta_1;\rho_2,\theta_2) = \rho_1+\rho_2+[(\rho\cos\theta-\rho_1\cos\theta_1-\rho_2\cos\theta_2)^2 + (\rho\sin\theta-\rho_1\sin\theta_1-\rho_2\sin\theta_2)^2]^{\frac{1}{2}} \tag{B 6}$$

and

$$Q_3'' = Q_3'\exp[l(f-\rho)].$$

Let us first examine the asymptotic behaviour for $l\to\infty$ of the double integral

$$I = \int\int_{-\pi}^{\pi} Q_3''(\rho,\theta;\rho_1,\theta_1;\rho_2,\theta_2)\,\mathrm{e}^{-lf(\rho,\theta;\rho_1,\theta_1;\rho_2,\theta_2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \tag{B 7}$$

Applying the Laplace method twice yields

$$I\to\frac{2\pi}{l}Q_3''(\rho,\theta;\rho_1,\theta;\rho_2,\theta)\left(\frac{\rho-\rho_1-\rho_2}{\rho\rho_1\rho_2}\right)^{\frac{1}{2}}\mathrm{e}^{-lf(\rho,\theta;\rho_1,\theta;\rho_2,\theta)}, \tag{B 8}$$

which shows that, to leading order, the infinite integration domain for ρ_1 and ρ_2 may be reduced to the domain over which $\rho-\rho_1-\rho_2 > 0$, where $f(\rho,\theta;\rho_1,\theta;\rho_2,\theta) = \rho$, and $Q_3'' = Q_3'$.

The next step is to employ the definition of the kernel Q_3' (2.25), which gives for $\theta = \theta_1 = \theta_2 = 0$ and $\rho = 1$

$$Q_3' = Q_3'' = (\epsilon l^3)^2 \left\{\frac{\rho_1[2\rho_2(1-\rho_2)-(1-\rho_1)+2\rho_2(1-\rho_1)]}{2(1-\rho_1)(1-\rho_2)(\rho_1+\rho_2)} + \frac{2\rho_2(1-\rho_1-\rho_2)}{(1-\rho_2)(\rho_1+\rho_2)}\right\}. \quad \text{(B 9)}$$

Finally, since

$$\int_0^1 \int_0^{1-\rho_1} Q_3'(1,0;\rho_1,0;\rho_2,0)\,(\rho_1\rho_2)^{\frac{1}{2}}(1-\rho_1-\rho_2)^{\frac{1}{2}}\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2 = \pi(\epsilon l^3)^2(\log 2-\tfrac{143}{210}), \quad \text{(B 10)}$$

we obtain from (B 5) and (B 8)

$$a_3(1,0) = \tfrac{1}{2}\epsilon^2 l^5(\log 2-\tfrac{143}{210}) \sim 6.1\times 10^{-3}\epsilon^2 l^5, \quad \text{(B 11)}$$

Again, similarly to the two-dimensional example, the three-dimensional third-order term yields a non-uniform contribution for $\epsilon l^{\frac{5}{2}} = \epsilon/Fr^5 = O(1)$. The numerical reduction compared with the lower orders, however, is even more drastic than for the two-dimensional case (B 4). Thus, by comparing a_3 (B 11) with a_2 (4.10) and a_1 (2.22), we get

$$\frac{a_2(1,0)}{a_1(1,0)} = 0.16\epsilon l^{\frac{5}{2}}; \quad \frac{a_3(1,0)}{a_2(1,0)} = 0.039\epsilon l^{\frac{5}{2}}; \quad \frac{a_3(1,0)}{a_1(1,0)+a_2(1,0)} = \frac{0.039\epsilon^2 l^5}{1+0.16\epsilon l^{\frac{5}{2}}}, \quad \text{(B 12)}$$

and the impact upon the wave drag (2.18) is small for $\epsilon l^{\frac{5}{2}} = O(1)$.

Concluding this Appendix, it has been found that the third-order term (2.24) of the regular perturbation expansion yields a small contribution to the amplitude of the far free waves and the wave drag in the range of non-uniformity investigated here. This has been found, however, for the particular pressure distributions selected in the text, and the general validity of this result is a matter of further investigations.

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